Principal Eigenvalues in Large Drift

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Motivation

A reaction-advection-diffusion equation

$$\begin{cases} \dfrac{\partial U}{\partial t} = \overbrace{\nabla \cdot (a \nabla U)}^{\text{Diffusion}} + \overbrace{Aq \cdot \nabla U}^{\text{Advection}} + \overbrace{f(x, y, U)}^{\text{Reaction}} & \text{in } \Omega, \\ n \cdot \nabla U = 0 & \text{on } \partial\Omega, \end{cases}$$

models the dynamics of a chemical of density $U(t, x, y)$ in a reactive medium with an advective field $q(x, y)$.

Why Principal Eigenvalue?

- In some cases $u(t, x, y)$ has the form $\phi(x \cdot e - ct, x, y)$ where c is the speed of propagation of the front ϕ.
- The main question is to understand the influence of large advection on the speed of propagation.
- The speed c has a formulation given via the principal eigenvalue of the linearizing operator.
- This leads to questions about the asymptotic behaviour of the principal eigenvalue when the amplitude A of the flow q goes to ∞.

Eigenvalues with Dirichlet Boundary Condition

We start with simple elliptic eigenvalue problems first.

$$\begin{cases} -\Delta\phi_A + A\,q\cdot\nabla\phi_A = \lambda_A\phi_A & \text{in} \quad \Omega, \\ \phi_A = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

- Ω is a bounded domain in $\mathbb{R}^N$ of class $C^2(\Omega)$, with an outward unit normal n.

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- Ω is a bounded domain in $\mathbb{R}^N$ of class $C^2(\Omega)$, with an outward unit normal n.
- q is an $L^\infty(\Omega)$ vector field such that $\int_\Omega \nabla\cdot q\,\phi = 0$, for all ϕ in $C_c^\infty(\Omega)$.

Eigenvalues with Dirichlet Boundary Condition

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- For all $A \in \mathbb{R}$, λ_A is the principal eigenvalue and ϕ_A is the principal eigenfunction.
- For each self-adjoint elliptic PDE, the principal eigenvalue is given by the variational formula involving the Rayleigh quotient

$$\lambda_A = \min_{\phi \in H_0^1(\Omega)} \frac{\int |\nabla\phi|^2}{\int \phi^2}.$$

First Integrals

It turns out that the asymptotic behaviour of λ_A depends on what we call "first integrals" of the flow q.

Definition

A function w is said to be a first integral of the vector field q if $w \in H^1(\Omega), w \neq 0$ and $q.\nabla w = 0$ a.e. in Ω. In other words, the streamlines of q are level sets of w.

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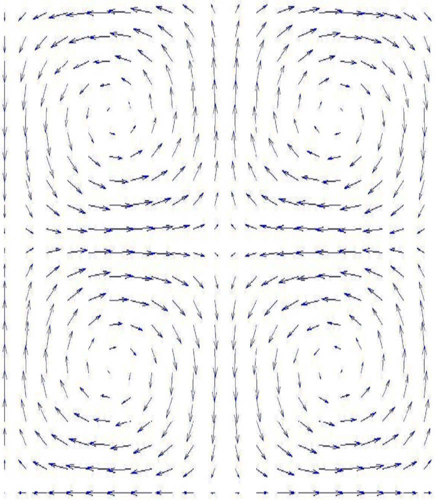
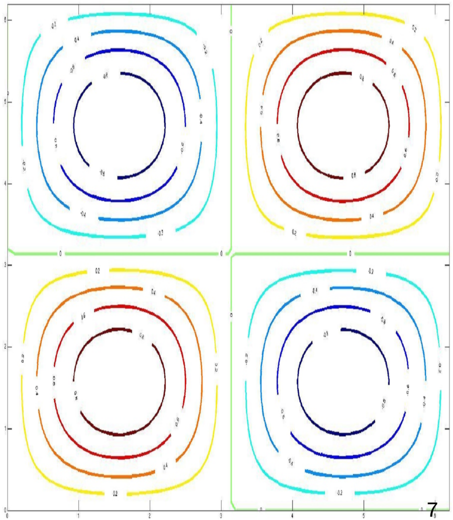
Notation

$$\mathcal{I}_0 = \{w |\ w \text{ is a first integral of } q \text{ and } w = 0 \text{ on } \partial\Omega\}.$$

Example

Let $q = \nabla^{\perp}\psi = (-\partial_y \psi, \partial_x \psi)$ be a two-dimensional flow. Say $\psi(x, y) = sin(x)sin(y)$. Then, clearly ψ is a first integral.





Theorem (Berestycki, Hamel, Nadirashvili (2005))

i) *If* $\mathcal{I}_0 \neq \emptyset$, *then* $\lim\limits_{A \to \infty} \lambda_A = \min\limits_{w \in \mathcal{I}_0} \dfrac{\int_\Omega |\nabla w|^2}{\int_\Omega w^2}$.

ii) *If* $\mathcal{I}_0 = \emptyset$, *then* $\lim\limits_{A \to \infty} \lambda_A = +\infty$.

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ii) *If* $\mathcal{I}_0 = \emptyset$, *then* $\lim\limits_{A\to\infty} \lambda_A = +\infty$.

Moreover, for all $A \in \mathbb{R}$ *and* $w \in \mathcal{I}_0$,

$$\lambda_A \leq \frac{\displaystyle\int_\Omega |\nabla w|^2}{\displaystyle\int_\Omega w^2}.$$

Proof

Step 1. If $\{\lambda_{A_n}\}$ be a bounded sequence, then there exist a subsequence $\{A_{n_k}\}$ and $w \in \mathcal{I}_0$ such that

$$\liminf_{A_{n_k} \to \infty} \lambda_{A_{nk}} \geq \frac{\displaystyle\int_\Omega |\nabla w|^2}{\displaystyle\int_\Omega w^2}.$$

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Step 2. If $\mathcal{I}_0 \neq \emptyset$, then

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Let $\{\lambda_{A_n}\}$ be bounded. Let's find a nonzero first integral.

$$-\Delta\phi_{A_n} + A_n\, q \cdot \nabla\phi_{A_n} = \lambda_{A_n}\phi_{A_n}, \tag{1}$$

where $\phi_{A_n} \in H_0^1(\Omega)$.

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$$\int_\Omega |\nabla\phi_{A_n}|^2 + \frac{A_n}{2}\underbrace{\int_\Omega q\cdot\nabla(\phi_{A_n}^2)}_{\text{is 0 since } \nabla\cdot q=0} = \lambda_{A_n}\int_\Omega \phi_{A_n}^2. \tag{2}$$

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Since ϕ_{A_n} is an eigenfunction, we can assume $\displaystyle\int_\Omega \phi_{A_n}^2 = 1$.

Recall

Rellich Theorem

Let Ω be a bdd domain in $\mathbb{R}^N$ which has a smooth boundary. Let $\{u_n\}$ be a family of functions in Ω such that $\{u_n\}$ and $\{\nabla u_n\}$ be uniformly bounded in $L^2(\Omega)$, then there exists a subsequence $\{u_{n_k}\}$ and $u \in H^1$ such that

$$\begin{array}{lll} u_{n_k} \to u & \text{in} & L^2(\Omega) \\ u_{n_k} \rightharpoonup u & \text{in} & H^1(\Omega). \end{array}$$

Moreover,

$$\liminf_{n_k \to \infty} \|\nabla u_{n_k}\|_{L^2(\Omega)} \geq \|\nabla u\|_{L^2(\Omega)}.$$

Step 1

Rellich theorem yields that there exist a subsequence $\{n_k\}$ and a function $w \in H_0^1(\Omega)$ such that

$$\begin{array}{rcl} \phi_{A_{n_k}} \to w & \text{in} & L^2(\Omega), \\ \phi_{A_{n_k}} \rightharpoonup w & \text{in} & H_0^1(\Omega), \\ \liminf_{A_{n_k}\to\infty} \left\| \nabla \phi_{A_{n_k}} \right\|_{L^2(\Omega)} & \geq & \|\nabla w\|_{L^2(\Omega)}. \end{array}$$

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$$\liminf_{A_{n_k}\to\infty} |\lambda_{A_{n_k}}| \geq \frac{\int_\Omega |\nabla w|^2}{\int_\Omega w^2}.$$

Now it's enough to show $q \cdot \nabla w = 0$.

Step 1

We know

$$-\Delta\phi_{A_{n_k}} + A_{n_k}\, q \cdot \nabla\phi_{A_{n_k}} = \lambda_{A_{n_k}}\phi_{A_{n_k}}, \tag{4}$$

devide both side by A_{n_k}, then

$$-\frac{1}{A_{n_k}}\Delta\phi_{A_{n_k}} + q \cdot \nabla\phi_{A_{n_k}} = \frac{\lambda_{A_{n_k}}}{A_{n_k}}\phi_{A_{n_k}},$$

now take limit when $A_{n_k} \to \infty$, so

$$0 = \lim_{A_{n_k}\to\infty} q \cdot \nabla\phi_{A_{n_k}} = q \cdot \nabla w,$$

so w is a first integral of flow q. ♣

Step 2: finding upper bound

Now let $w \in \mathcal{I}_0$. If ϕ is the eigenfunction corresponding to λ,

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$$\int_\Omega \nabla\phi \cdot \nabla \frac{w^2}{\phi + \varepsilon} + \cancelto{0}{A \int_\Omega q \cdot \nabla(\ln(\phi + \varepsilon) w^2)} - \cancelto{0}{A \int_\Omega q \cdot \nabla w^2 (\phi + \varepsilon)} =$$

$$\lambda \int_\Omega \frac{\phi}{\phi + \varepsilon} w^2.$$

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$\int_\Omega \nabla\phi \cdot \nabla \frac{w^2}{\phi+\varepsilon} = \lambda \int_\Omega \frac{\phi}{\phi+\varepsilon} w^2$. But $\int_\Omega \nabla\phi \cdot \nabla \frac{w^2}{\phi+\varepsilon} \leq \int_\Omega |\nabla w|^2$

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$$\int_\Omega \nabla\phi\cdot\nabla\left(\frac{w^2}{\phi+\varepsilon}\right) = \int_\Omega \frac{2w(\phi+\varepsilon)\nabla\phi\cdot\nabla w - w^2|\nabla\phi|^2}{(\phi+\varepsilon)^2}$$

$$= \underbrace{-\int_\Omega \frac{(w\nabla\phi - w(\phi+\varepsilon))\cdot(w\nabla\phi - w(\phi+\varepsilon))}{(\phi+\varepsilon)^2}}_{\text{is always } \le 0} + \int_\Omega |\nabla w|^2$$

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So

$$\lambda\int_\Omega \frac{\phi}{\phi+\varepsilon}w^2 \le \int_\Omega |\nabla w|^2$$

Send $\varepsilon\to 0$, then

$$0 \le \lambda_A \le \frac{\int_\Omega |\nabla w|^2}{\int_\Omega w^2}.\clubsuit$$

Extension to More General Elliptic Problems

Here we discuss the case of an Elliptic PDE with Dirichlet boundary condition.

$$\begin{cases} -\nabla \cdot (a\nabla\phi_A) + Aq.\nabla\phi_A + C\phi_A = \lambda_A P\phi_A & \text{in } \Omega, \\ \phi_A = 0 & \text{on } \partial\Omega, \end{cases}$$

where $a(x) = (a_{ij}(x))$ is a $C^1(\Omega)$ symmetric matrix and there exist positive numbers θ and β, such that

$$\theta|\xi|^2 \leq \sum_{1\leq i,j\leq N} a_{ij}(x)\xi_i\xi_j \leq \beta|\xi|^2$$

- There exist two positive numbers p_1 and p_2 such that $p_1 \leq P \leq p_2$.
- $C(x) \in L^\infty(\Omega)$

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Theorem (Berestycki, Hamel, Nadirashvili (2005))

1. *If $\mathcal{I}_0 \neq \emptyset$, then λ_A is bounded*

$$\lambda_A \to \min_{w \in \mathcal{I}_0} \frac{\int_\Omega \nabla w \cdot a(x)\nabla w + C(x)w^2}{\int_\Omega P w^2} \text{ as } A \to \infty.$$

2. *If $\mathcal{I}_0 = \emptyset$, then $\lambda_A \to \infty$ as $A \to \infty$.*

Elliptic PDE With Neumann Boundary Condition

$$\begin{cases} -\nabla \cdot (a\nabla\phi_A) + A\,q.\nabla\phi_A + C\phi_A = \lambda_A\phi_A & \text{in } \Omega, \\ n \cdot \nabla\phi_A = 0 & \text{on } \partial\Omega. \end{cases}$$

All assumptions are same except some changes about vector field q, $\nabla \cdot q = 0$ a.e. in Ω and $q \cdot n = 0$ in $L^1_{loc}(\partial\Omega)$.

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λ_A is bounded and

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Why is $q \cdot n = 0$ necessary?

We give a counterexample which shows if $q \cdot n \neq 0$, then theorem does not hold.

Example

$$\begin{cases} -\phi_A'' + A\phi_A' + c(x)\phi_A = \lambda_A \phi_A \quad \text{in } (0,1), \\ \phi_A'(0) = \phi_A'(1) = 0. \end{cases}$$

Here, $q = 1$ and $q \cdot n \neq 0$. First integrals are nonzero constants.

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- Now, let $c \neq 0$ be a continuous function such that $c(0) < \int_0^1 c(x)dx$. We see that theorem does not hold in this case.

Why is $q \cdot n = 0$ necessary?

First we rerwite equation in a self-adjoint way,

$$\begin{cases} -(e^{-Ax}\phi_A^{'})^{'} + c(x)e^{-Ax}\phi_A = \lambda_A e^{-Ax}\phi_A & \text{in } (0,1) \\ \phi_A^{'}(0) = \phi_A^{'}(1) = 0. \end{cases}$$

So

$$\lambda_A = \min_{\phi \in H^1(0,1)} \frac{\int_0^1 e^{-Ax}\phi'^2 + c(x)e^{-Ax}\phi_A^2}{\int_0^1 e^{-Ax}\phi^2} \leq \frac{\int_0^1 c(x)e^{-Ax}}{\int_0^1 e^{-Ax}}.$$

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But $c(x)$ is a continous function in $[0,1]$. So according to Stone-Weierstrass theorem, it can be approximated uniformly by a sequence $\{P_n(x)\}$ of polynomials.

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$$\begin{aligned} \frac{\int_0^1 c(x)e^{-Ax}}{\int_0^1 e^{-Ax}} &= \lim_{n\to\infty} \frac{\int_0^1 (a_0 + a_1 x + \ldots + a_n x^n)e^{-Ax}}{\int_0^1 e^{-Ax}}. \\ &= c(0) + \lim_{n\to\infty} \frac{\int_0^1 (a_1 x + \ldots + a_n x^n)e^{-Ax}}{\int_0^1 e^{-Ax}}. \end{aligned}$$

Why is $q.n = 0$ necessary?

so $\lambda_A \leq c(0)$. According to theorem

$$\lambda_A = \min_{\phi \in \mathcal{I}_0} \frac{\int_0^1 e^{-Ax}\phi'^2 + c(x)e^{-Ax}\phi^2}{\int_0^1 e^{-Ax}\phi^2} = \int_0^1 c(x).$$

So $\int_0^1 c(x) \leq c(0)$, which contradicts with assumption. ♣

Parabolic framework

$$\begin{cases} u_t^A = \Delta u^A - A\, q.\nabla u^A & t > 0, \\ u^A(t,.) = 0 & \text{on } \partial\Omega \text{ and } t \geq 0, \\ u^A(0,.) = u_0(.). & \end{cases}$$

Theorem (Berestycki, Hamel, Nadirashvili (2005))

The following properties are equivalent;

i) *There exists $u_0 \in H_0^1(\Omega)$ such that $\lim\limits_{A\to\infty} u^A(1,.) \neq 0$.*

ii) *The vector field q has a nonzero first integral in $H_0^1(\Omega)$.*

iii) *$\{\lambda_A\}$ is bounded as $A \to \infty$.*

Proof from (ii) to (i)

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$$\begin{aligned} \phi_n &\to w \quad \text{in} \quad L^2(\Omega), \\ \phi_n &\rightharpoonup w \quad \text{in} \quad H_0^1(\Omega). \end{aligned}$$

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Let u_n and $k_n = e^{-\lambda_n t}\phi_n$ be the solution of (22) with initial conditions w and ϕ_n in order. Call $h_n = u_n(t,.) - e^{-\lambda_n t}\phi_n(.)$

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$$\begin{cases} (h_n)_t = \Delta h_n - Aq \cdot \nabla h_n, & t > 0 \\ h_n(t,.) = 0 & \text{on } \partial\Omega, t \geq 0 \\ h_n(0,.) = w(.) - \phi_n(.) & \end{cases}$$

Proof from (ii) to (i)

Since on the RHS it is an elliptic equation with Dirichlet boundary condition so from first theorem iii and ii are equivalent.
From proof of first theorem there exist a sequence $A_n \to \infty$ and $w \in \mathcal{I}_0$ such that

$$\begin{aligned} \phi_n &\to w \quad \text{in} \quad L^2(\Omega), \\ \phi_n &\rightharpoonup w \quad \text{in} \quad H_0^1(\Omega). \end{aligned}$$

Let u_n and $k_n = e^{-\lambda_n t}\phi_n$ be the solution of (22) with initial conditions w and ϕ_n in order. Call $h_n = u_n(t,.) - e^{-\lambda_n t}\phi_n(.)$ so h_n is a solution of

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Multiply equation by h_n and integrate by parts

Proof from (ii) to (i)

$$\int_\Omega \int_{t_1}^{t_2} (h_n)_t h_n = \int_{t_1}^{t_2} \int_\Omega \Delta h_n h_n - \underbrace{\frac{A_n}{2} \int_{t_1}^{t_2} \int_\Omega q \cdot \nabla h_n^2}_{\text{is 0 since } \nabla \cdot q = 0}$$

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\frac{1}{2} \int_\Omega (h_n^2(t_2,.) - h_n^2(t_1,.))dx &= \underbrace{- \int_{t_1}^{t_2} \|\nabla h_n\|^2_{L^2(\Omega)}\, dt}_{\text{is always } \leq 0}
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so for each $t_1 < t_2$ we have

$$\|h_n(t_2,.)\|^2_{L^2(\Omega)} \leq \|h_n(t_1,.)\|^2_{L^2(\Omega)} . \tag{5}$$

Now let $t_1 = 0$ & $t_2 = 1$,

$$\|h_n(1,\cdot)\|^2_{L^2(\Omega)} \leq \|w - \phi_n\|^2_{L^2(\Omega)} .$$

Proof from (ii) to (i)

Now let $n \to \infty$, so we get $\lim_{n\to\infty} \|h_n(1,.)\|^2_{L^2(\Omega)} = 0$. But $h_n(1,.) = u_n(1,.) - e^{\lambda_n}\phi_n(.)$, so

$$\lim_{n\to\infty} u_n(1,.) = \exp\Big(\min_{w\in\mathcal{I}_0(\Omega)} \frac{\int |\nabla w|^2}{\int w^2} \Big) w \neq 0. \clubsuit$$

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Remark

from the inequality (5) it's clear not only for t=1, but also in each finite time t_0 the result is true, meaning $\lim_{n\to\infty} u_n(t_0,.) \neq 0$.

Proof from (i) to (ii) According to our assumption, there exists an initial function $u_0 \in H_0^1(\Omega) \setminus \{0\}$ such that $u_n(1,.) \nrightarrow 0$ in $L^2(\Omega)$ as $n \to \infty$.
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[1] There exist $\varepsilon > 0$, $M > 0$ and a sequence $A_n \longrightarrow \infty$, such that

$$\begin{aligned} &0 \le u_0 \le M && \text{a.e.} \\ &0 \le u_n(t,\cdot) \le M && \text{a.e.} \\ &\|u_n(1,\cdot)\|_{L^2(\Omega)} \ge \varepsilon && \text{for all } A_n, \end{aligned}$$

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[3] We will prove the function w that we found in step 2, at a specific time, is a nonzero first integral in $H_0^1(\Omega)$.

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Rellich theorem yeilds that there is a subsequence $\{u_{n_k}\}$ and a function w_1 in $H_0^1((0,1)\times\Omega)$ such that

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Now Let's prove $q\cdot\nabla w = 0$,

$$\lim_{A_{n_k}\to\infty} \frac{1}{A_{n_k}} u_t^{A_{n_k}} = \lim_{A_{n_k}\to\infty} \left(\frac{1}{A_{n_k}}\Delta u^{A_{n_k}} - q\cdot\nabla u^{A_{n_k}}\right).$$

So,

$$0 = \lim_{A_{n_k}\to\infty} q.\nabla u^{A_{n_k}} = q\cdot\nabla w.$$

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So w is not zero in $(0,1)\times\Omega$.
To sum up, for almost every $t \in (0,1)$, the function $w(t,.)$ is in $H_0^1(\Omega)$ and satisfies $q(.)\cdot\nabla w(t,.) = 0$ a.e. in Ω. From (8), one concludes there exists at least a $t_0 \in (0,1)$ such that $w(t_0,.)$ is a nonzero first integral of q in $H_0^1(\Omega)$. ♣

Conclusion

WE studied the asymptotic behaviour of the principal eigenvalue of some linear elliptic or parabolic PDE with large advection, in the case of an incompressible flow.

We saw this behaviour is dircetly related to the first integrals of underlying velocity field q.

- If there is a nonzero first integral the sequence of principal eigenvalues are going to be bounded.
- If there is no nonzero first integral, the sequence goes to infinity.